St George Girls High School

Year 12

Common Test 3

2009

Mathematics Extension 1

General Instructions
- Working time – 75 minutes
- Reading time – 5 minutes
- Write using blue or black pen
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.
- Write on one side of the page only.
- Start each question on a new page.

Total marks – 66
- Attempt Questions 1 – 6
- All questions are of equal value

Question	Mark
Question 1	/11
Question 2	/11
Question 3	/11
Question 4	/11
Question 5	/11
Question 6	/11
Total	/66

Students are advised that this is a School Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Question 1 – Start a New Page – (11 marks)

Marks

a) If α, β, γ are the zeros of the polynomial $P(x) = 2x^3 + 8x^2 - x + 6$ evaluate:

(i) $\alpha + \beta + \gamma$ 1

(ii) $\alpha\beta + \beta\gamma + \alpha\gamma$ 1

(iii) $\alpha\beta\gamma$ 1

(iv) $\alpha^2 + \beta^2 + \gamma^2$ 2

b) By drawing a suitable graph, solve $x^2(x-1)(x+2) \geq 0$ 2

c) When $Q(x) = ax^3 + bx^2 + c$ is divided by $(x+2)$ the remainder is 3 and, when $Q(x)$ is divided by $(x^2 - 1)$ the remainder is $(2x + 4)$. 4

Find a, b and c.

Question 2 – Start a New Page – (11 marks) Marks

a) Evaluate: $\int_0^{\frac{\pi}{3}} \sin^2 2x \ dx$ 2

b) The polynomials $3x^3 - x + 1$ and $ax(x-1)(x+2) + bx(x-1) + cx + d$ are equal for [redacted] all values of x. 4

Determine the values of a, b, c and d.

c) (i) Express $2\sin x + \sqrt{12}\cos x$ in the form $R\sin(x+\theta)$ where $R > 0$ and θ is a subsidiary angle in the range $0 \leq \theta \leq \frac{\pi}{2}$ 5

(ii) Hence, give the general solution to the equation

$$2\sin x + \sqrt{12}\cos x = 2\sqrt{3}$$

Marks

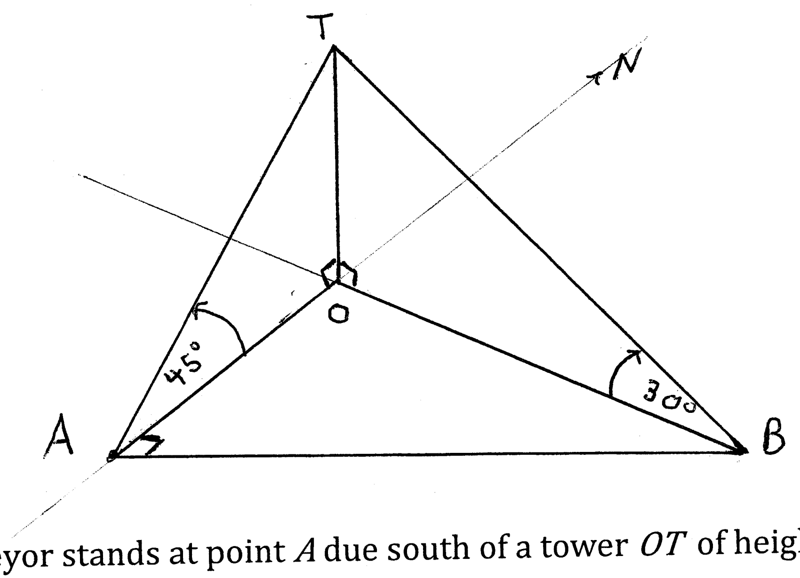
Question 3 – Start a New Page – (11 marks)

a) Evaluate: $\int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1-4x^2}}$ 2

b) Sketch the graph of $y = sin^{-1}\left(\frac{x}{2}\right)$. Clearly indicate the domain and range on your graph. 3

c) Differentiate $(1 + x^2).\tan^{-1} x$ 2

d) 4



A surveyor stands at point A due south of a tower OT of height h metres.

The angle of elevation of the top of the tower from A is 45°. The surveyor then walks 100 metres due east to point B, from here the angle of elevation to the top of the tower is 30°.

(i) Show that $h = 50\sqrt{2}$

(ii) Calculate the bearing of B from the base of the tower.

Question 4 – Start a New Page – (11 marks)

Marks

a) Evaluate: $\sin^{-1}\left(\cos\frac{\pi}{3}\right)$ 2

b) Use the table of standard integrals to show that $\int_0^{15} \frac{dx}{\sqrt{x^2+64}} = 2\ln 2$ 2

c) (i) Show that $\frac{u}{u+1} = 1 - \frac{1}{u+1}$ 1

(ii) Hence, find $\int \frac{dx}{1+\sqrt{x}}$ using the substitution $x = u^2 \ [u \geq 0]$ 3

d) Given that $y = \sin^{-1}(\sqrt{x})$, show that $\frac{dy}{dx} = \frac{1}{\sin 2y}$ 3

Marks

Question 5 – Start a New Page – (11 marks)

a) Show that $(2x + 1)$ is a factor of $2x^3 + 7x^2 - x - 2$ **1**

b) (i) Show that $\sin x - \cos 2x = 2\sin^2 x + \sin x - 1$ **2**

(ii) Hence, or otherwise, solve $\sin x - \cos 2x = 0$ for $0 \le x \le 2\pi$ **3**

c) If $a.\cos x = 1 + \sin x$

(i) prove that $\frac{a-1}{a+1} = t$, where $t = \tan\frac{x}{2}$ **3**

(ii) hence, solve $1 + \sin x = 2\cos x$ for $0^\circ < x < 360^\circ$, to nearest degree. **2**

Marks

Question 6 – Start a New Page – (11 marks)

a) Evaluate: $\int_1^{e^3} \frac{dx}{x(9+(\ln x)^2)}$ using the substitution $u = \ln x$ 4

b) A monic polynomial $P(x)$ of degree 4 is known to have exactly two zeros at 2 and -2. It is also known that $P(x)$ is an even function. 3

Further, when $x = 3$ the value of $P(x)$ is 55. Determine the polynomial function $P(x)$.

c) Consider $\tan^{-1}y = 2\tan^{-1}x$. 4

(i) Express y as a function of x

(ii) Show that the function has no turning point.

End of Paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1

a) $P(x) = 2x^3 + 8x^2 - x + 6$

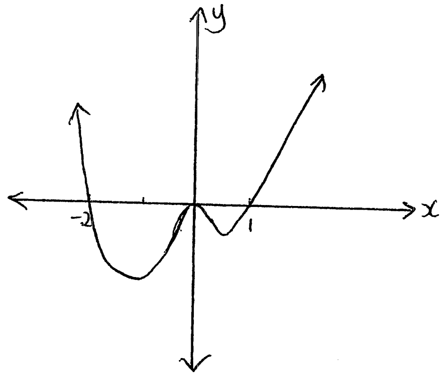
(i) $\alpha + \beta + \gamma = -\frac{b}{a}$
$= -\frac{8}{2}$
$= -4$

(ii) $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$
$= -\frac{1}{2}$

(iii) $\alpha\beta\gamma = -\frac{d}{a}$
$= -\frac{6}{2}$
$= -3$

(iv) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$
$= (-4)^2 - 2(-\frac{1}{2})$
$= 16 + 1$
$= 17$

b)



$x \le -2$, $x \ge 1$ and $x = 0$

c) $Q(x) = ax^3 + bx^2 + c$

$Q(-2) = 3$

$a(-2)^3 + b(-2)^2 + c = 3$

$-8a + 4b + c = 3$ ①

$Q(1) = 2x + 4$

$a(1)^3 + b(1)^2 + c = 2 + 4$

$a + b + c = 6$ ②

$Q(-1) = 2x + 4$

$a(-1)^3 + b(-1)^2 + c = 2$

$-a + b + c = 2$ ③

② - ③ we get

$a + b + c = 6$ ②
$-a + b + c = 2$ ③

$2a = 4$
$\therefore a = 2$

Subst. $a = 2$ into ① we get

$-8(2) + 4b + c = 3$
$-16 + 4b + c = 3$
$4b + c = 19$ ④

Subst $a = 2$ into ② we get

$2 + b + c = 6$
$\therefore b + c = 4$ ⑤

④ - ⑤ we get

$4b + c = 19$
$b + c = 4$

$3b = 15$
$\therefore b = 5.$

Subst. $b = 5$ into ⑤ we obtain

$5 + c = 4$
$\therefore c = -1$

So $a = 2$, $b = 5$, $c = -1$

Question 2

a) $\int_0^{\frac{\pi}{3}} \sin^2 2x \, dx$

$= \int_0^{\frac{\pi}{3}} \frac{1}{2}(1-\cos 4x)\,dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{3}} (1-\cos 4x)\,dx$

$= \frac{1}{2}\left[x - \frac{\sin 4x}{4}\right]_0^{\frac{\pi}{3}}$

$= \frac{1}{2}\left\{\left[\frac{\pi}{3} - \frac{1}{4}\sin\frac{4\pi}{3}\right] - \left[0 - \frac{1}{4}\sin 0\right]\right\}$

$= \frac{1}{2}\left\{\left[\frac{\pi}{3} - \frac{1}{4}\times -\frac{\sqrt{3}}{2}\right] - [0]\right\}$

$= \frac{1}{2}\left[\frac{\pi}{3} + \frac{\sqrt{3}}{8}\right]$

b) $3x^3 - x + 1 \equiv ax(x-1)(x+2) + bx(x-1) + cx + d$

Equating

coefficient of x^3 $\quad 3 = a$

when $x=0$ $\quad 1 = d$

$x=1$ $\quad 3 = c+d$

So $3 = c+1$

$\therefore c = 2$

Try when $x=-2$,

$3(-2)^3 - (-2) + 1 \equiv 0 + b(-2)(-2-1) + c(-2) + d$

$-24 + 2 + 1 \equiv 6b - 2c + d$

$-21 \equiv 6b - 2\times 2 + 1$

$-21 \equiv 6b - 4 + 1$

$-21 \equiv 6b - 3$

$6b \equiv -18$

$\therefore b = -3$

So $a = 3$, $b = -3$, $c = 2$, $d = 1$

9 (i). $2\sin x + \sqrt{12}\cos x$ $\qquad R\sin(x+\theta)$, $R>0$

$R\sin(x+\theta) = R\sin x\cos\theta + R\cos x\sin\theta$

so $2\sin x + \sqrt{12}\cos x = R\sin x\cos\theta + R\cos x\sin\theta$

Equating coefficients of $\sin x$, $\qquad R\cos\theta = 2 \quad (1)$

Equating coefficients of $\cos x$ $\qquad R\sin\theta = \sqrt{12} \quad (2)$

Squaring and adding (1) and (2)

$$R^2\cos^2\theta + R^2\sin^2\theta = 4+12$$

$$R^2(\sin^2\theta + \cos^2\theta) = 16$$

$$\therefore R^2 = 16$$

$$R = \pm\sqrt{16} \quad \text{but } R>0$$

$$\text{so } R = 4$$

From (1), $\qquad 4\cos\theta = 2$

$$\cos\theta = \frac{1}{2}$$

$$\therefore \theta = \frac{\pi}{3}$$

From (2) $\qquad R\sin\theta = \sqrt{12}$

$$4\sin\theta = \sqrt{4}\sqrt{3}$$

$$4\sin\theta = 2\sqrt{3}$$

$$\sin\theta = \frac{\sqrt{3}}{2}$$

$$\therefore \theta = \frac{\pi}{3}$$

So θ is in the first quadrant with related angle $\frac{\pi}{3}$

So $2\sin x + \sqrt{12}\cos x = 4\sin\left(x + \frac{\pi}{3}\right)$.

(ii) $2\sin x + \sqrt{12}\cos x = 2\sqrt{3}$

then $4\sin\left(x+\frac{\pi}{3}\right) = 2\sqrt{3}$

$$\sin\left(x+\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\sin\left(x+\frac{\pi}{3}\right) = \sin\frac{\pi}{3}$$

ie $\quad x + \frac{\pi}{3} = 2n\pi + \frac{\pi}{3}$ $\qquad$ OR $\quad x = (2n+1)\pi - \frac{\pi}{3}$

$x = 2n\pi$ $\qquad x = 2n\pi + \pi - \frac{\pi}{3}$

$$x = 2n\pi + \frac{2\pi}{3}$$

Question 3

a) $\displaystyle\int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1-4x^2}}$

$\displaystyle = \int_0^{\frac{1}{4}} \frac{1}{\sqrt{1-4x^2}}\, dx$

$\displaystyle = \int_0^{\frac{1}{4}} \frac{1}{\sqrt{4(\frac{1}{4}-x^2)}}$

$\displaystyle = \int_0^{\frac{1}{4}} \frac{1}{2\sqrt{\frac{1}{4}-x^2}}$

$\displaystyle = \frac{1}{2}\int_0^{\frac{1}{4}} \frac{1}{\sqrt{\frac{1}{4}-x^2}}\, dx$

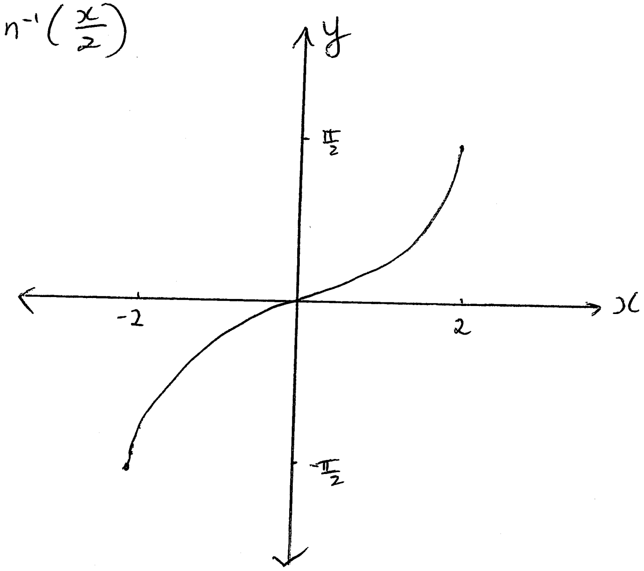
$\displaystyle = \frac{1}{2}\int_0^{\frac{1}{4}} \frac{1}{\sqrt{(\frac{1}{2})^2-x^2}}\, dx$

$\displaystyle = \frac{1}{2}\left[\sin^{-1}\left(\frac{x}{\frac{1}{2}}\right)\right]_0^{\frac{1}{4}}$

$\displaystyle = \frac{1}{2}\left[\sin^{-1}(2x)\right]_0^{\frac{1}{4}}$

$\displaystyle = \frac{1}{2}\left[\sin^{-1}(\tfrac{1}{2}) - \sin^{-1}(0)\right] = \frac{1}{2}\left[\frac{\pi}{6} - 0\right] = \frac{\pi}{12}$

b) $y = \sin^{-1}\left(\frac{x}{2}\right)$



$y = \sin^{-1} x$ has domain $-1 \le x \le 1$

so $y = \sin^{-1}(\frac{x}{2})$ has domain $-1 \le \frac{x}{2} \le 1$

$\therefore$ Domain $-2 \le x \le 2$

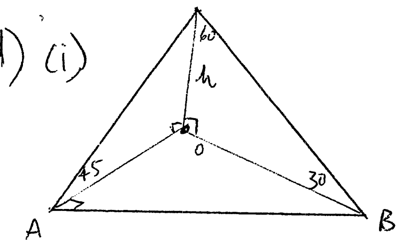
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

c) $\displaystyle \frac{d}{dx}\left[(1+x^2).\tan^{-1}x\right]$

$\displaystyle = \tan^{-1}x \,.\, 2x + (1+x^2).\frac{1}{1+x^2}$

$= 2x\tan^{-1}x + 1$

d) (i)



In $\triangle OAT$

$\tan 45° = \frac{h}{OA}$

$h = OA \tan 45$

$h = OA \times 1$

$\therefore h = OA$

In $\triangle OBT$,

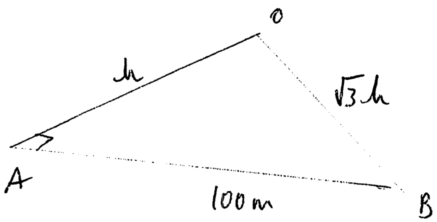
$\tan 30 = \frac{h}{OB}$

$\therefore h = OB \tan 30°$

$h = OB \times \frac{1}{\sqrt{3}}$

$h = \frac{OB}{\sqrt{3}}$

$OB = \sqrt{3}h$



$(\sqrt{3}h)^2 = h^2 + 100^2$

$3h^2 = h^2 + 100^2$

$2h^2 = 100^2$

$h^2 = \frac{100^2}{2}$

$h = \frac{\sqrt{100^2}}{\sqrt{2}}$ $\quad \therefore h = \frac{100}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{100\sqrt{2}}{2} = \underline{\underline{50\sqrt{2}}}$

d) (ii) $\tan \angle AOB = \frac{100}{50\sqrt{2}}$

$= \frac{2}{\sqrt{2}}$

$= \frac{2\sqrt{2}}{2}$

$\tan \angle AOB = \sqrt{2}$

$\therefore \angle AOB = 54° 44'$

So B is $(180° - 54° 44')$ from O

so bearing is $\underline{125° 16'}$

Question 4

a) $\sin^{-1}\left(\cos\frac{\pi}{3}\right)$

$= \sin^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{6}$

b) show $\displaystyle\int_0^{15} \frac{dx}{\sqrt{x^2+64}} = 2\ln 2$

$$\int_0^{15} \frac{1}{\sqrt{x^2+8^2}}\,dx = \left[\ln\left[x + \sqrt{x^2+8^2}\right]\right]_0^{15}$$

$$= \left[\ln\left[15 + \sqrt{15^2+64}\right]\right] - \left[\ln\left[0 + \sqrt{0^2+8^2}\right]\right]$$

$$= \ln\left[15 + \sqrt{289}\right] - \ln\sqrt{64}$$

$$= \ln\left[15+17\right] - \ln 8$$

$$= \ln 32 - \ln 8$$

$$= \ln\left(\frac{32}{8}\right)$$

$$= \ln 4$$

$$= \ln 2^2$$

$$= 2\ln 2 \text{ as required.}$$

c) (i) Show that $\dfrac{u}{u+1} = 1 - \dfrac{1}{u+1}$

LHS $\dfrac{u+1-1}{u+1}$

$$= \frac{u+1}{u+1} - \frac{1}{u+1}$$

$$= 1 - \frac{1}{u+1}$$

$$= \text{RHS}.$$

(ii) Find $\displaystyle\int \frac{dx}{1+\sqrt{x}}$ using substitution $x = u^2$ $(u \geq 0)$

$x = u^2$

$\frac{dx}{du} = 2u$

$dx = 2u\,du$

$$= \int \frac{1}{1+\sqrt{u^2}} \cdot 2u\,du$$

$$= \int \frac{1}{1+u} \cdot 2u\,du$$

$$= \int \frac{2u}{1+u} \cdot du$$

$$= 2\int \frac{u}{u+1}\,du$$

(from above) $= 2\displaystyle\int \left(1 - \frac{1}{u+1}\right) du$

$$= 2\left[u - \ln(u+1)\right] + C$$

$$= 2u - \ln(u+1)^2 + C$$

d) $y = \sin^{-1}(\sqrt{x})$ Show that $\frac{dy}{dx} = \frac{1}{\sin 2y}$

$y = \sin^{-1}(x^{\frac{1}{2}})$ — ①

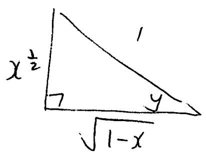
Then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}}$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}\sqrt{1-x}}$

Now $\frac{1}{\sin 2y} = \frac{1}{2\sin y \cos y}$

From ① $\sin y = x^{\frac{1}{2}}$

$\cos y = \sqrt{1-x}$



$\therefore \frac{1}{\sin 2y} = \frac{1}{2\sqrt{x}\sqrt{1-x}}$

$= \frac{dy}{dx}$

OR $\sqrt{x} = \sin y$

$x = \sin^2 y$

$\frac{dx}{dy} = 2\sin y \cos y$

$\frac{dy}{dx} = \frac{1}{2\sin y \cos y} = \frac{1}{\sin 2y}$.

Question 5

a) If $(2x+1)$ is a factor, $P(-\frac{1}{2}) = 0$

$2(-\frac{1}{2})^3 + 7(-\frac{1}{2})^2 - (-\frac{1}{2}) - 2$

$= -\frac{1}{4} + \frac{7}{4} + \frac{1}{2} - 2$

$= 0.$

b) (i) Show that $\sin x - \cos 2x = 2\sin^2 x + \sin x - 1$

LHS.
We know
$\cos 2x = 1 - 2\sin^2 x$

so $\sin x - \cos 2x$

$= \sin x - (1 - 2\sin^2 x)$

$= \sin x - 1 + 2\sin^2 x$

$= 2\sin^2 x + \sin x - 1$

$=$ R.H.S.

(iii). Solve $\sin x - \cos 2x = 0$ for $0 \le x \le 2\pi$

Let $m = \sin x$

$\therefore 2\sin^2 x + \sin x - 1 = 0$

ie $2m^2 + m - 1 = 0$

$2m^2 + 2m - m - 1 = 0$

$2m(m+1) - (m+1) = 0$

$(2m-1)(m+1) = 0$

$\therefore m = \frac{1}{2}$ OR $m = -1$

Replacing for m

$\sin x = \frac{1}{2}$ OR $\sin x = -1$

$\therefore x = \frac{\pi}{6}$ OR $\pi - \frac{\pi}{6}$ OR

$x = \frac{\pi}{6}$, $\frac{5\pi}{6}$ OR $\frac{3\pi}{2}$

c) If $a. \cos x = 1 + \sin x$

(i) Prove that $\frac{a-1}{a+1} = t$, where $t = \tan\frac{x}{2}$.

If $t = \tan\frac{x}{2}$

then $\sin x = \frac{2t}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$

so $a. \cos x = 1 + \sin x$

$a.\left(\frac{1-t^2}{1+t^2}\right) = 1 + \left(\frac{2t}{1+t^2}\right)$

$a\left(\frac{1-t^2}{1+t^2}\right) = \frac{1+t^2}{1+t^2} + \frac{2t}{1+t^2}$

$a(1-t^2) = t^2 + 2t + 1$

$a(1-t^2) = (t+1)^2$

$a(1-t)\cancel{(1+t)} = (1+\cancel{t})^2$

$a(1-t) = (1+t)$

$a - at = 1 + t$

$a - 1 = t + at$

$a - 1 = t(1+a)$

$\therefore t = \frac{a-1}{a+1}$ as required.

(ii) Hence, solve $1+\sin x = 2\cos x$ for $0° < x < 360°$ to nearest degree.

$1+\sin x = a\cos x$

↑

then $\underline{a=2}$

and since $t = \frac{a-1}{a+1}$,

$t = \frac{2-1}{2+1}$

$t = \frac{1}{3}$.

where $t = \tan\frac{x}{2}$,

$\frac{1}{3} = \tan\frac{x}{2}$

$\frac{x}{2} = \tan^{-1}\left(\frac{1}{3}\right)$

$\frac{x}{2} = 18°26'$

$\therefore x = 36° 52' 11.63''$

so $\underline{x = 37°}$ (to nearest degree).

Question 6

a) $\displaystyle\int_1^{e^3} \frac{dx}{x(9+(\ln x)^2)}$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x}\,dx$.

$\therefore dx = x\,du$.

when $x = e^3$, $\underline{u=3}$

$x = 1$, $\underline{u=0}$.

$= \displaystyle\int_0^3 \frac{du}{9+u^2}$

$= \frac{1}{3}\left[\tan^{-1}\frac{u}{3}\right]_0^3$

$= \frac{1}{3}\left[\tan^{-1} 1 - \tan^{-1} 0\right]$

$= \frac{1}{3}\left[\frac{\pi}{4} - 0\right]$

$= \frac{\pi}{12}$

b) $P(x) = x^4 + bx^3 + cx^2 + dx + e$

Since $P(x)$ is even

Then $P(x) = x^4 + cx^2 + e$.

$P(2) = 0$

$\therefore 2^4 + c(2)^2 + e = 0$

$16 + 4c + e = 0$

$4c + e = -16$ ①

When $x = 3$, $P(x)$ is 55,

$3^4 + c(3)^2 + e = 55$

$81 + 9c + e = 55$

$9c + e = -26$ ②

② − ① we have:

$5c = -10$

$\therefore c = -2$

When $c = -2$, subst into ① we have

$4(-2) + e = -16$

$-8 + e = -16$

$e = -16 + 8$

$e = -8$

$\therefore P(x) = x^4 - 2x^2 - 8$

c) $\tan^{-1} y = 2\tan^{-1} x$

Let $\tan^{-1} x = \theta$ so $\tan\theta = x$

then $\tan^{-1}(y) = 2\theta$

$\therefore y = \tan 2\theta$

$y = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

since $\tan\theta = x$, $y = \dfrac{2x}{1 - x^2}$.

(ii) Show that the function has no turning point.

$\dfrac{dy}{dx} = \dfrac{(1-x^2).2 - 2x(-2x)}{(1-x^2)^2}$

$= \dfrac{2 - 2x^2 + 4x^2}{(1-x^2)^2}$

$\dfrac{dy}{dx} = \dfrac{2x^2 + 2}{(1-x^2)^2}$

for turning pts, $\dfrac{dy}{dx} = 0$,

$\therefore 0 = 2x^2 + 2$

$0 = 2(1 + x^2)$

$x^2 + 1 = 0$

$x^2 = -1$

$\therefore$ no solution and $\therefore$ no turning points.

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